Matrix methods in the analysis of complex networks

Random walks

Dario Fasino

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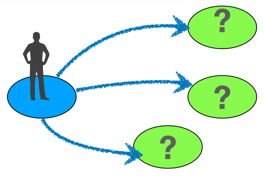
Introduction

Random walk on a graph $\mathcal{G} = (V, E)$:

- Start from a node $i \in V$
  (possibly chosen with some probability)
    - Pick one of the outgoing edges
    - move to the destination of the edge
    - Repeat.



Applications

Random walks are widely used in network science to

- model user navigation and epidemic spreading
- quantify node centrality and accessibility
- reveal network communities and core-periphery structures.

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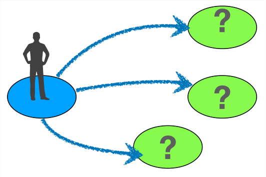

From here onward $\mathcal{G} = (V, E)$ denotes a directed graph without sink nodes: $d_i^{out} > 0$ for $i \in V$.

Questions

- Where is the walker after k steps?
- Long-term behavior? Depends on the initial node?
- How many timesteps to go from i to j?

Notations

- X_k: position of the walker at time $k = 0, 1, 2 \dots$ (random variable)
- $x(k)$: probability vector, $x(k)_i = \mathbb{P}(X_k = i)$
- P: column stochastic matrix, $P_{ij} = \mathbb{P}(X_{k+1} = i | X_k = j)$ (independently of k)

A random walk is a sequence of probability vectors $\{x(k)\}$ such that $x(k+1) = Px(k)$ where the transition matrix P is column stochastic.

$$x(k) = P^k x(0).$$

In fact, $(P^k)_{ij} = \mathbb{P}(X_{\ell+k} = i | X_\ell = j)$.

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Let A be the adjacency of a possibly weighted digraph,
$A_{ij} =$ weight of $(i, j) \in E$ if present, $A_{ij} = 0$ else.
Then the transition matrix of the associated RW is

$$P_{ij} = \frac{A_{ji}}{\sum_{\ell=1}^{n} A_{j\ell}}.$$

If the graph is not weighted then the random walk is uniform.

Ergodicity

A probability vector π is stationary for P if $P\pi = \pi$. At least one is definitely there.

The random walk is ergodic if there is only one stationary probability vector π and for all initial probability vectors x_0 the random walk converges to π:

$$\lim_{k\to\infty} x(k) = \pi.$$

Equivalently,

$$\lim_{k\to\infty} P^k = \pi e^{\mathrm{T}}.$$

Lemma

A random walk is ergodic iff P is aperiodic and $\rho(P)$ is simple. Moreover, if P is irreducible then $\pi > 0$.

Proof: From P-F theory.

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Theorem

Let $\mathcal{G}'$ be the subgraph of $\mathcal{G}$ made by all globally reachable nodes. The random walk on $\mathcal{G}$ is ergodic iff $\mathcal{G}'$ is not empty and aperiodic. Moreover, if $\mathcal{G}$ is strongly connected then $\pi > 0$.

Proof: See Enrico's second lecture.

PageRank

Let P be the stochastic matrix describing the RW on a graph $\mathcal{G}$,
let $v > 0$ be a fixed probability vector, and $\alpha \in (0, 1)$.

The matrix $G = \alpha P + (1 - \alpha) v e^{\mathrm{T}}$ is the Google matrix of the graph $\mathcal{G}$.

The Brin-Page navigation model

At each time step, the walker

- with probability α performs a step according to the RW rule;
- with probability $1 - \alpha$ jumps to a node j chosen with probability v_j.

The matrix G is positive and stochastic $\rightsquigarrow$ The RW associated with G is ergodic, the stationary vector π is positive, and

$$\begin{cases} G\pi = \pi \\ e^T \pi = 1 \end{cases} \qquad \rightsquigarrow \qquad (I - \alpha P)\pi = (1 - \alpha) v.$$

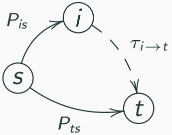
Problem

What is the average number of timesteps for the walker to go from node s to node t?

Hitting time of t starting from s: $\tau_{s\to t} = \mathbb{E}(k|X_k = t, X_0 = s)$.

If $s = t$ then $\tau_{s\to t} = 0$, otherwise

$$\begin{aligned}\tau_{s\to t} &= P_{ts}\cdot 1 + \textstyle\sum_{i\neq t} P_{is}(\tau_{i\to t}+1)\\ &= 1 + \textstyle\sum_{i=1}^{n} P_{is}\tau_{i\to t}.\end{aligned}$$



Hitting time matrix

The hitting time matrix $T = (\tau_{i\to j})$ solves the equation

$$(I - P^{\mathrm{T}})T = ee^{\mathrm{T}} - \mathrm{Diag}(\tau_1,\dots,\tau_n)$$

where $\tau_i = \mathbb{E}(k > 0|X_k = i, X_0 = i)$ is the return time of node i.

Hitting times and return times

Let P be irreducible, $\pi = (\pi_1, \ldots, \pi_n)^{\mathrm{T}} > 0$ be the stationary probability vector,

$$P\pi = \pi, \qquad e^{\mathrm{T}}\pi = 1.$$

From $(I - P^{\mathrm{T}})T = ee^{\mathrm{T}} - \mathrm{Diag}(\tau_1, \ldots, \tau_n)$ we obtain

$$\begin{aligned} 0 = \pi^{\mathrm{T}}(I - P^{\mathrm{T}})T &= \pi^{\mathrm{T}}(ee^{\mathrm{T}} - \mathrm{Diag}(\tau_1, \ldots, \tau_n)) \\ &= e^{\mathrm{T}} - \pi^{\mathrm{T}}\mathrm{Diag}(\tau_1, \ldots, \tau_n). \end{aligned}$$

Kac's lemma

For an irreducible RW with stationary prob. vector $\pi > 0$,

$$\tau_i = 1/\pi_i.$$

Thus $\mathrm{Diag}(\tau_1, \ldots, \tau_n) = \mathrm{Diag}(\pi_1, \ldots, \pi_n)^{-1}$.

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From $(I - P^{\mathrm{T}})T = ee^{\mathrm{T}} - \mathrm{Diag}(\tau_1, \ldots, \tau_n)$ we obtain

$$\begin{aligned}(I - P^{\mathrm{T}})T\pi &= (ee^{\mathrm{T}} - \mathrm{Diag}(\tau_1, \ldots, \tau_n))\pi \\ &= e - \mathrm{Diag}(\pi_1, \ldots, \pi_n)^{-1}\pi = 0.\end{aligned}$$

But then $T\pi \in \mathrm{Ker}(I - P^{\mathrm{T}}) = \langle e \rangle$.

Random target lemma

$$(T\pi)_i = \sum_{j=1}^{n} \pi_j \tau_{i\to j} = \kappa,$$

where $\kappa > 0$ is the Kemeny's constant.

A special case

Let $\mathcal{G}$ be irreducible and undirected, $A = A^T$.

Then $P = A\,\mathrm{Diag}(d)^{-1}$ and $\pi = d/(e^{\mathrm{T}}d)$ where $d = Ae$ is the degree vector.

Define $M = (A/(e^{\mathrm{T}}d) - \mathrm{Diag}(\pi) - \pi\pi^{\mathrm{T}})^{-1}$ and $m = \mathrm{diag}(M)$.
Note that $M = M^{\mathrm{T}}$.

Then $T = M - em^{\mathrm{T}}$.

- The symmetric part $\frac{1}{2}(T + T^{\mathrm{T}}) = M$ and $M_{ij} = \tau_{i\to j} + \tau_{j\to i}$.
- The skew-symmetric part $\frac{1}{2}(T - T^{\mathrm{T}}) = \frac{1}{2}(me^{\mathrm{T}} - em^{\mathrm{T}})$, thus

$$m_i - m_j = 2(\tau_{i\to j} - \tau_{j\to i}).$$

  The number m_i is the RW centrality of $i \in V$.

J. D. Noh, H. Rieger. Random walks on complex networks. *Phys. Rev. Lett.* 92 (2004), 118701.

Ergodicity coefficients

Let $\mathcal{S} = \{x \geq 0, e^{\mathrm{T}}x = 1\}$ be the set of probability n-vectors.

Let $\|\cdot\|_p$ be a vector norm in $\mathbb{R}^n$. For any column stochastic matrix P the number

$$\tau_p(P) = \sup_{x,y\in\mathcal{S}} \frac{\|P(x-y)\|_p}{\|x-y\|_p}$$

is an ergodicity coefficient. A notable case:

The Dobrushin coefficient

$$\tau_1(P) = \sup_{x,y\in\mathcal{S}} \frac{\|P(x-y)\|_1}{\|x-y\|_1} = \sup_{e^{\mathrm{T}}v=0} \frac{\|Pv\|_1}{\|v\|_1}.$$

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Explicit formula: If P is stochastic then

$$\tau_1(P) = \frac{1}{2} \max_{j,k} \sum_{i=1}^{n} |P_{ij} - P_{ik}|.$$

Ergodicity coefficients - properties

Let P, P' be column stochastic matrices.

- $0 \leq \tau_1(P) \leq 1$. Moreover, $\tau_1(P) = 0$ if and only if $\mathrm{rank}(P) = 1$.
- $|\tau_1(P) - \tau_1(P')| \leq \|P - P'\|_1$.
- $\tau_1(\alpha P + (1-\alpha)P') \leq \alpha\tau_1(P) + (1-\alpha)\tau_1(P')$.
- $\tau_1(PP') \leq \tau_1(P)\tau_1(P')$.

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Theorem

Let P be a stochastic matrix. If $\tau_1(P) < 1$ then the Markov chain associated to P is ergodic:
The stochastic solution of $x = Px$ is unique and

$$\|x_k - x\|_1 \leq \tau_1(P)^k \|x_0 - x\|_1.$$

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For the Google matrix $G = \alpha P + (1-\alpha)ve^{\mathrm{T}}$ we have

$$\begin{aligned}\tau_1(G) &= \tau_1(\alpha P + (1-\alpha)ve^{\mathrm{T}}) \\ &\leq \alpha\tau_1(P) + (1-\alpha)\tau_1(ve^{\mathrm{T}}) \leq \alpha.\end{aligned}$$

Hence, for the Brin-Page navigation model we have ergodicity and

$$\|x(k) - \pi\|_1 \leq \alpha^k \|x(0) - \pi\|_1.$$

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Theorem

Let P, P' be irreducible, stochastic matrices. If $Px = x$ and $P'x' = x'$ (stationary prob. vectors) then

$$\|x - x'\|_1 \leq \frac{\|P - P'\|_1}{1 - \tau_1(P)}.$$